Beauty Of Mathematics

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Abstract— **It is note that contemporary mathematics is cornerstone of engineering science and technology and vice versa. It turns out that there exist enormous number of universal formulas for taking definite integrals by hand. It is shown that square-root differential operator gives rise the definite probabilistic measure. Complicated second order differential equations depending many parameters are considered. Fractional derivatives are also played an important role in mechanics and engineering.**

Keywords Infinite number of universal formulas for taking definite integrals. Mellin representation, square-root differential operator, probabilistic measure, differential equations for harmonic oscillator type phenomena, fractional derivatives.

I. INTRODUCTION

For all time, pure human brain product-mathematics plays a vital role in our civilization. Developments of natural science, engineering science and technology give rise absolutely new branch of mathematics and vice versa. Simple example is that the Dirac delta function $\delta(x)$,

$$\int_{-\infty}^{\infty} dx f(x)\delta(x) = f(0) \qquad (1)$$

arisen from quantum mechanics [1,2,3] is base of generalized functions and functional integrals [4,5] which are widely used in quantum physics [6,7] and the string theory [8,9,10]. Other example is that the group theory [11], especially $U(3)\times SU(2)$ – gauge group [12] gives rize the quantum field theory [13,14], in particular it's the standard model [15,16].

The Lorentz transformations, vector and tensor [17] calculations in the curvilinear spacetime are base of the Einstein theory of special and general theories of gravitation [18].

At the same time, geometric [19] and algebraical theories play an important role in the axiomatic approach in the quantum field theory [20] and even more in the proof of the Ferma math theorem [21].

Contemporary fine instrumental laser, high technology and engineering science allow us to carry out very high precision measurements in physics. Striking examples are that the detection of gravitational waves [22,23], correlation of two photons interactions at large distances (action at distances) named entanglement process [24,25] and very fine accuracy measurements of leptons anomalous magnetic moments [26,27]. So that all these are the best experimental measurements and theories physicists have ever done.

Here, notice that disagreement $a_\mu^{exp} - a_\mu^{sm}$ between theoretical calculations in the standard model and the experimental data for the muon anomalous magnetic moment is easily solved within the framework of the nonlocal theory and the existences of photino [28].

Even more, we believe that in near future the Hawking thermal radiation [29,30] will be detected, which may be played an important role in understanding structure of the universe in deeply. In this review paper we consider some beauties of mathematics.

II. BEAUTY OF INTEGRAL CALCULUS

Since from Isaac Newton and Gottfried W.Leibniz's time integral and differential calculus play a vital role in science and human knowledge about nature and society. It is well known that taking integrals encounter some difficulties with respect to differentiation with respect to arbitrary functions. However, it turns out that accidentally there exist an enormous number of universal formulas for taking integrals for sign variables functions, like *sinx, cosx,* *exp*($-x^n$) and etc. Recently, by using the Mellin representation, we have obtained around three hundred universal formulas [31,32] for taking integrals by hand. Here, there are some examples:

a. Universal formula 1

$$I_1 = \int_0^\infty dx\, x^\gamma \sin^m(bx^\nu) =$$

$$\frac{1}{2^{m-1}} N_m\left(\xi = -\frac{1}{2}\left[1+\frac{1+\gamma}{\nu}\right]\right)\left(\frac{2}{b}\right)^{\frac{1+\gamma}{\nu}}$$

$$\times \frac{\sqrt{\pi}}{2\nu} \frac{\Gamma\left[\frac{1}{2}\left(1+\frac{1+\gamma}{\nu}\right)\right]}{\Gamma\left(1-\frac{1+\gamma}{2\nu}\right)}, \qquad (2)$$

where

$$N_1 = 1, N_3(\xi) = 3 - 3^{1+2\xi},$$

$$N_5(\xi) = 5^{1+2\xi} - 5\cdot 3^{1+2\xi} + 10$$

$$N_7(\xi) = -7^{1+2\xi} + 7\cdot 5^{1+2\xi} - 21\cdot 3^{1+2\xi} + 35$$

etc.

1. $$\int_0^\infty dx\, \frac{\sin bx}{\sqrt{x}} = \sqrt{\frac{\pi}{2b}}. \qquad (3)$$

2. $$\int_0^\infty dx\, \frac{\sin^7 bx}{\sqrt{x}} = \frac{1}{64}\sqrt{\frac{\pi}{2b}}\left(-\sqrt{7}+7\sqrt{5}-21\sqrt{3}+35\right). \qquad (4)$$

3. $$\int_0^\infty dx\, \frac{\sin^3 bx}{x^{19/20}} = \frac{\sqrt{\pi}}{8}\left(\frac{2}{b}\right)^{-9/10} \frac{\Gamma\left(\frac{1}{20}\right)}{\Gamma\left(\frac{29}{20}\right)}\left(3-3^{9/10}\right) \qquad (5)$$

4. $$\int_0^\infty dx\, \sin(bx^2) = \frac{1}{2}\sqrt{\frac{\pi}{2b}} \qquad (6)$$

5. $$\int_0^\infty dx\, \frac{\sin bx^{80}}{x^{41}} = \frac{1}{40}\sqrt{\frac{\pi b}{2}} \qquad (7)$$

6. $$\int_0^\infty dx\, \frac{\sin^7 bx^{-1}}{x} = -\frac{5\pi}{32} \qquad (8)$$

7. $$\int_0^\infty dx\, \frac{\sin bx^{60}}{x^{21}} = \frac{\sqrt{\pi}}{120}\left(\frac{2}{b}\right)^{-1/3} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{7}{6}\right)} \qquad (9)$$

8. $$\int_0^\infty dx\, \frac{\sin^7 bx^5}{x} = \frac{\pi}{32} \qquad (10)$$

b. Universal formula 2

$$I_2 = \int_0^1 dx\,(1-x^\sigma)^\mu \sin^m\left[b(1-x^\sigma)^\lambda\right]$$

$$= \frac{b}{2^{m-1}}\frac{1}{2i\sigma}\int_{-\beta+i\infty}^{-\beta-i\infty} d\xi\, \frac{b^{2\xi}}{\sin\pi\xi\,\Gamma(2+2\xi)} N_m(\xi) \times$$

$$B\left(\frac{1}{\sigma}, 1+\mu+\lambda+2\lambda\xi\right), \qquad (11)$$

where $m = 1,3,5\ldots, -1 < \beta < 0$.

9. $$\int_0^\infty dx\,(1-x^2)\sin^m\left[b(1-x^2)^{-1/2}\right]$$

$$= \frac{\pi b}{2}\frac{1}{2^m}\left[N_m(0) - \frac{1}{3}b^2 N_m(1) + \frac{1}{12}b^3 N_m\left(\frac{3}{2}\right)\right] +$$

$$\frac{b\sqrt{\pi}}{2^m}\sum_{n=2}^{\infty}(-1)^n \frac{b^{2n+1}}{(2+2n)!\left(\frac{1}{2}-n\right)} \times$$

$$n\,\frac{\Gamma\left(\frac{1}{2}+n\right)}{n!} N_m\left(n+\frac{1}{2}\right) \qquad (12)$$

c. Universal formula 3

$$I_3 = \int_0^\infty dx\, x^\gamma J_\sigma(bx^\nu) =$$

$$\frac{1}{2\nu}\left(\frac{b}{2}\right)^{-\frac{1+\gamma}{\nu}} \frac{\Gamma\left(\frac{1+\gamma+\nu\sigma}{2\nu}\right)}{\Gamma\left(1+\sigma-\frac{1+\gamma+\nu\sigma}{2\nu}\right)},$$

$$b > 0 \qquad (13)$$

10. $$\int_0^\infty dx\, x^{-1} J_2(bx^2) = \frac{1}{4} \qquad (14)$$

11. $$\int_0^\infty dx\, \frac{1}{x^2} J_2(bx^2) = \frac{4\pi}{5}\sqrt{b}\,\frac{1}{\Gamma^2\left(\frac{1}{4}\right)}, \quad b > 0 \qquad (15)$$

d. Universal formula 4

$$I_4 = \int_0^\infty dx\, x^\gamma \left(e^{-bx^\rho}\right)^{e^{ax^\nu}} =$$

$$\frac{1}{\nu}\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} d\xi \frac{\Gamma\left(\frac{1+\gamma+\rho\xi}{\nu}\right)}{sin\pi\xi\Gamma(1+\xi)} \times$$

$$b^\xi[-a\xi]^{-\frac{1+\gamma+\rho\xi}{\nu}} \qquad (16)$$

12. $$\int_0^\infty dx \left(e^{-bx^2}\right)^{e^{ax^2}} =$$

$$\frac{1}{2\sqrt{b}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma\left(\frac{1}{2}+n\right)\left[\frac{a\left(n+\frac{1}{2}\right)}{b}\right]^n, \qquad (17)$$

where limit $a \to 0$ gives Gaussian famous integral:

$$\int_0^\infty dx\ e^{-bx^2} = \frac{1}{2}\sqrt{\frac{\pi}{b}} \qquad (18)$$

as it should be.

Finally, notice that, at present time, we have calculate around 1450 concrete integrals of above types [33].

III. BEAUTY OF SQUARE ROOT DIFFERENTIAL OPERATOR

e. It turns out that the square-root differential operator is reduced the definite probabilistic measure [34]:

$$\omega(\rho) = \frac{1}{\pi}\frac{1}{\sqrt{1-\rho^2}} \qquad (19)$$

with well-known properties:

$$\int_{-1}^{1} d\rho\ \ \omega(\rho) = 1, \qquad (20)$$

$$\int_{-1}^{1} d\rho\ \ \rho\omega(\rho) = 0, \qquad (21)$$

$$\int_{-1}^{1} d\rho\ \ \rho^2\omega(\rho) = \frac{1}{2}, \qquad (22)$$

Thus the Green function for the Weyl equation [35]

$$\sqrt{m^2 - \Box}\, W^c(x) = \delta^4(x),$$

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \boldsymbol{x}^2}$$

has two equivalent solutions

$$W^c_{1sp}(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}} \times$$

$$\frac{1}{(2\pi)^4}\int d^4p\, e^{-ipx}\frac{m+\hat{p}\rho}{m^2-p^2\rho^2-i\varepsilon}. \qquad (23)$$

and

$$W^c_{2sp}(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}} \times$$

$$\frac{1}{(2\pi)^4}\int d^4p\, e^{-ipx}\frac{m\rho+\hat{p}}{m^2\rho^2-p^2-i\varepsilon} \qquad (24)$$

Here energy value for these two cases takes the form:

$$\omega_1 = \frac{1}{|\rho|}\sqrt{m^2+\vec{p}^2\rho^2} \qquad (25)$$

and

$$\omega_1 = \sqrt{m^2\rho^2+\vec{p}^2} \qquad (26)$$

respectively .

The second case (26) gives a finite value for an averaged energy

$$\langle\omega_2\rangle = \frac{2}{\pi}\int_0^1 d\rho \frac{1}{\sqrt{1-\rho^2}}\ \sqrt{m^2\rho^2+\vec{p}^2} =$$

$$\frac{2\sqrt{m^2+\vec{p}^2}}{\pi}E\left(\frac{\pi}{2},\frac{m}{\sqrt{m^2+\vec{p}^2}}\right) \qquad (27)$$

and

$$\langle\omega_2^2\rangle = \frac{1}{2}m^2 + \vec{p}^2.$$

Here $E\left(\frac{\pi}{2},\frac{m}{\sqrt{m^2+\vec{p}^2}}\right)$ is the elliptic integral of the second kind [36]:

$$E(\varphi,k) = \int_0^\varphi \sqrt{1-k^2sin^2\alpha}\, d\alpha \qquad (28)$$

We assume that if there exist some new force which is carried by a square root particle, then a potential of this force is given by formula:

$$U_D(r) = \frac{\lambda}{(2\pi)^3}\int d^3p\, e^{i\vec{p}\vec{r}}\frac{1}{\sqrt{m^2+\vec{p}^2}} =$$

$$\frac{\lambda}{2\pi^2}\left(\frac{m}{r}\right)K_1(mr), \qquad (29)$$

where $K_1(z)$ is the Macdonald function and λ is some coupling constant. Asymptotic behavior of this potentine takes the form:

$$U_D(r) = \begin{cases} \frac{\lambda}{4\pi^2} m^2 ln \frac{C \cdot z}{2}, z = mr \to 0 \\ \frac{\lambda}{4\pi^2} \frac{m^2}{z} \sqrt{\frac{\pi}{2z}}\, e^{-z}, z = mr \to \infty \end{cases} \qquad (30)$$

$C = 0.57721566490 \ldots$

f. Now let us consider the square-root differential equation:

$$\sqrt{a^2 + \frac{d^2}{dt^2}}\, X(t) = 0, \qquad (31)$$

where the probabilistic measure (19) appears:

$$\frac{1}{\sqrt{a^2 + \frac{d^2}{dt^2}}} f(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{d\rho}{\sqrt{1-\rho^2}} \times$$

$$\int_0^\infty d\alpha\, e^{-\alpha\left(a^2 + \rho^2 \frac{d^2}{dt^2}\right)} \times$$

$$\left(a + i\rho \frac{d}{dt}\right) f(t). \qquad (32)$$

Here

$$e^{-\alpha\rho^2 \frac{d^2}{dt^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \alpha^n \rho^{2n} \frac{d^{2n}}{dt^{2n}} \qquad (33)$$

and we use the following integrals:

$$\frac{1}{\pi} \int_{-1}^{1} \frac{d\rho}{\sqrt{1-\rho^2}} \rho^{2n} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n+1)}, \qquad (34)$$

$$\int_0^\infty d\alpha\, \alpha^n\, e^{-\alpha\, a^2} = (\, a^2)^{-1-n} \Gamma(1+n), \qquad (35)$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{d\rho}{\sqrt{1-\rho^2}} \rho^{1+2n} = 0. \qquad (36)$$

Then, we have nice formula

$$\hat{D} = \frac{1}{\sqrt{a^2 + \frac{d^2}{dt^2}}}$$

where

$$\hat{D} f(t) = \frac{1}{\sqrt{a^2 + \frac{d^2}{dt^2}}} f(t) =$$

$$\frac{1}{a\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(n + \frac{1}{2}\right)}{n!} \frac{1}{a^{2n}} \left(\frac{d^2}{dt^2}\right)^n f(t). \qquad (3$$

In particular:

$$\hat{D}C = \frac{1}{a} C, \quad \hat{D}t = \frac{1}{a} t,$$

$$\hat{D}t^2 = \frac{1}{a} t^2 - \frac{1}{a^3},$$

$$\hat{D} sinbt = \Lambda(a,b) sinbt,$$

$$\hat{D} cosbt = \Lambda(a,b) cosbt,$$

$$\hat{D} e^{ibt} = \Lambda(a,b) e^{ibt},$$

$$\hat{D} e^{-ibt} = \Lambda(a,b) e^{-ibt},$$

$$\hat{D} e^{bt} = \Lambda'(a,b) e^{bt},$$

$$\hat{D} e^{-bt} = \Lambda'(a,b) e^{-bt},$$

and so on. Here

$$\Lambda(a,b) = \frac{1}{a\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)}{n!} \left(\frac{b^2}{a^2}\right)^n$$

$$= \frac{1}{a} \left(1 - \frac{b^2}{a^2}\right)^{-1/2} \qquad (38)$$

$$\Lambda'(a,b) = \frac{1}{a\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(n + \frac{1}{2}\right)}{n!} \left(\frac{b^2}{a^2}\right)^n$$

$$= \frac{1}{a} \left(1 + \frac{b^2}{a^2}\right)^{-1/2} \qquad (39)$$

So that

$$\hat{D} sinbt = \frac{sinbt}{\sqrt{a^2 - b^2}}$$

$$\hat{D} cosbt = \frac{cosbt}{\sqrt{a^2 - b^2}},$$

Finally, equation (31) takes the form:

$$\hat{N} X(t) = \sqrt{a^2 + \frac{d^2}{dt^2}}\, X(t) =$$

$$\frac{\left(a^2 + \frac{d^2}{dt^2}\right)}{\sqrt{a^2 + \frac{d^2}{dt^2}}} X(t) =$$

$$\left(a^2 + \frac{d^2}{dt^2}\right) \hat{D} X(t) =$$

$$\hat{D} \left(a^2 + \frac{d^2}{dt^2}\right) X(t) \qquad (40)$$

In particular:

$$\sqrt{a^2+\frac{d^2}{dt^2}}sinbt = \frac{a^2-b^2}{\sqrt{a^2-b^2}}sinbt. \quad (41)$$

$$\sqrt{a^2+\frac{d^2}{dt^2}}cosbt = \frac{a^2-b^2}{\sqrt{a^2-b^2}}cosbt \quad (42)$$

Therefore, the square-root differential equation

$$\sqrt{a^2+\frac{d^2}{dt^2}}X(t)=0 \quad (43)$$

describes also harmonic oscillator $X(t)=Asinat$ due to filter properties of the probabilistic measure (19)-(22).

Generalization of the equation (43):

$$\sqrt{m^2-\square}G_c(x)=\delta^4(x) \quad (44)$$

or

$$\frac{m^2-\square}{\sqrt{m^2-\square}}G_c(x)=\delta^4(x) \quad (45)$$

$$\left(\square=-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial \vec{x}^2}\right)$$

leads to the description of the generalized causal Green function for square-root Klein-Gordon equation, where

$$G_c(x)=\frac{1}{\sqrt{m^2-\square}}\delta^4(x)= \frac{1}{m\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma\left(n+\frac{1}{2}\right)}{n!\quad m^{2n}}\square^n\delta^4(x) \quad (46)$$

is the particular case of Efimov's nonlocal or generalized function [37] describing a nonlocal or extended object [38]. This object is distributed in a domain determined by the length:

$$L=\frac{\hbar}{mc}$$

It is obviously that the plane wave $\varphi(x)=\frac{1}{(2\pi)^{3/2}}\;e^{ipx}(px=p_0x^0-\vec{p}\vec{x})$ satisfies the square-root differential equation

$$\sqrt{m^2-\square}\,e^{ipx}=\frac{m^2-p^2}{\sqrt{m^2-p^2}}\;e^{ipx}=0 \quad (47)$$

if $m^2=p_0^2-\vec{p}^2$, where we have used the formula (46) with the change $\delta^4(x)\Longrightarrow\; e^{ipx}$.

IV. BEAUTY OF DIFFERENTIAL EQUATIONS

Beauty of mathematics allows us to obtain second order differential equations describing different oscillator processes.

g. First kind of differential equation is

$$\ddot{y}_1(t)+2b\dot{y}_1(t)+(b^2+\omega^2)y_1(t)=0 \quad (48)$$

Solution of which is given by

$$y_1(t)=A\;\;e^{-bt}sin\omega t. \quad (49)$$

When $b=0$ case is reduced the harmonic oscillator equation:

$$\ddot{y}_1(t)+\omega^2y_1(t)=0, \quad (50)$$

where

$$y_1(t)=A\;sin\omega t$$

as it should be.

h. Let us consider following differential equation:

$$\ddot{y}_2(t)+4bt\dot{y}_2(t)+ (4b^2t^2+2b+\omega^2)y_2(t)=0 \quad (51)$$

It is obviously that the case $b=0$ leads to again oscillator equation. Therefor the general solution of this equation is given by

$$y_2(t)=A\;\;e^{-bt^2}sin\omega t, \quad (52)$$

which describes the exponential damping oscillation process.

i. The case $b\to -b$ leads to the equation:

$$\ddot{y}_3(t)-4bt\dot{y}_3(t)+(4b^2t^2-2b+\omega^2)y_3(t)=0, \quad (53)$$

Solution of which is obviously:

$$y_3(t)=A\;\;e^{bt^2}sin\omega t \quad (54)$$

describing the exponential increasing oscillator process.

j. Lets see the following equation:

$$\ddot{y}_4(t)-2b\rho t^{\rho-1}\dot{y}_4(t)- \{b\rho t^{\rho-2}[\rho-1-b\rho t^{\rho}]-\omega^2\}y_4(t)=0 \quad (55)$$

solution of which is

$$y_4(t)=A\;\;e^{bt^{\rho}}sin\omega t. \quad (56)$$

k. For the general case, we have the following differential equation:

$$\ddot{y}_G(t) - 2\omega \frac{d}{dt} lnf \cdot \dot{y}_G(t) =$$

$$y_G(t)\left\{-2\omega\left[\frac{d}{dt} lnf\right]^2 + \frac{\ddot{f}}{f} - \omega^2\right\}, \qquad (57)$$

solution of which has the general form:

$$y_G(t) = A\, f(t) sin\omega t. \qquad (58)$$

l. Let us consider more complicated differential equation:

$$\ddot{y}_5(t) - 2\left(\frac{\rho}{t} + \frac{\nu}{(a^2+b^2t^\lambda)} \cdot \frac{\lambda b^2}{t} t^\lambda\right)\dot{y}_5(t) =$$

$$\left\{\frac{\nu\lambda b^2 t^\lambda}{t^\lambda(a^2+b^2t^\lambda)}(\lambda - 1 - 2\rho) - \frac{\nu\lambda^2 b^4 t^{2\lambda}}{t^2(a^2+b^2t^\lambda)^2}(1+\nu) - \right.$$

$$\left.\frac{\rho}{t^2}(1+\rho) - \omega^2\right\} y_5(t), \qquad (59)$$

solution of which is given by the formula:

$$y_5(t) = A\, t^\rho \left(a^2 + b^2 t^\lambda\right)^\nu sin\omega t \qquad (60)$$

1) for $\rho = \nu = 0$ case leads to the equation

$$\ddot{y}_5(t) - \omega^2 y_5(t) = 0 \qquad (61)$$

It has the solution

$$y_5(t) = A\, sin\omega t.$$

2) for $\rho = 0$, we have equation

$$\ddot{y}_6(t) - 2\frac{\nu}{a^2+b^2t^\lambda} \cdot \frac{\lambda b^2}{t}\, t^\lambda \dot{y}_6(t) =$$

$$\left[\frac{\nu\lambda b^2 t^\lambda}{t^2(a^2+b^2t^\lambda)}(\lambda - 1) - \right.$$

$$\left.\frac{\nu\lambda^2 b^4 t^{2\lambda}}{t^2(a^2+b^2t^\lambda)^2}(1+\nu) - \omega^2\right] y_6(t) \qquad (62)$$

solution of which is given by

$$y_6(t) = A\left(a^2 + b^2 t^\lambda\right)^\nu sin\omega t. \qquad (63)$$

3) The case $\nu = 0$ leads to the equation

$$\ddot{y}_7(t) - 2\frac{\rho}{t}\dot{y}_7(t) +$$

$$\left[\frac{\rho}{t^2}(1+\rho) + \omega^2\right] y_7(t) = 0 \qquad (64)$$

which has the solution:

$$y_7(t) = A\, t^\rho sin\omega t. \qquad (65)$$

4) The case $b = 0$ gives the equation:

$$\ddot{y}_8(t) - 2\frac{\rho}{t}\dot{y}_8(t) =$$

$$\left[-\frac{\rho}{t^2}(1+\rho) - \omega^2\right] y_8(t) = 0 \qquad (66)$$

which has the following solution:

$$y_8(t) = A\, t^\rho \cdot a^{2\nu} sin\omega t. \qquad (67)$$

5) The case $a = 0$ gives

$$\ddot{y}_9(t) - 2\left(\frac{\rho}{t} + \frac{\nu\lambda}{t}\right)\dot{y}_9(t) =$$

$$\left[\frac{\nu\lambda}{t^2}(\lambda - 1 - 2\rho) - \frac{\nu\lambda^2}{t^2}(1+\nu) \right.$$

$$\left. -\frac{\rho}{t^2}(1+\rho) - \omega^2\right] y_9(t) \qquad (68)$$

This equation has the solution:

$$y_9(t) = A\, t^\rho \cdot b^{2\nu} t^{\nu\lambda}\, sin\omega t. \qquad (69)$$

6) Finally for $\lambda = 0$ we have

$$\ddot{y}_{10}(t) - 2\frac{\rho}{t}\dot{y}_{10}(t) =$$

$$\left(-\frac{\rho}{t^2}(1+\rho) - \omega^2\right) y_{10}(t) \qquad (70)$$

This equation has the following solution

$$y_{10}(t) = A\, t^\rho (a^2 + b^2)^\nu\, sin\omega t \qquad (71)$$

m. In conclusion, we consider the following differential equation for sine and cosine functions:

$$\ddot{y}_{11}(t) - 2\nu sin\nu t \cdot \frac{1}{cos\nu t}\dot{y}_{11}(t) =$$

$$\left[2\nu^2 \frac{sin^2\nu t}{cos^2\nu t} - \nu^2 - \omega^2\right] y_{11}(t), \qquad (72)$$

solution of which is given by the formula

$$y_{11}(t) = A\, cos\nu t \cdot\, sin\omega t \qquad (73)$$

V. BEAUTY OF FRACTIONAL DERIVATIVES

n. Let us consider the following fractional derivatives of some elementary functions (see, also [39]):

$$\left(\frac{d}{dx}\right)^\rho\; C = 0,$$

$$\left(\frac{d}{dx}\right)^\rho\; sin(ax) = a^\rho sin\left(ax + \frac{\pi}{2}\rho\right),$$

$$\left(\frac{d}{dx}\right)^\rho\; cos(ax) = a^\rho cos\left(ax + \frac{\pi}{2}\rho\right),$$

$$\left(\frac{d}{dx}\right)^\rho\; e^{ax} = a^\rho e^{ax},$$

$$\left(\frac{d}{dx}\right)^\rho\; e^{-ax} = -a^\rho e^{-ax},$$

$$\left(\frac{d}{dx}\right)^\rho\; a^x = (lna)^\rho a^x, \qquad a > 1,$$

and so on. Here ρ is any number, even more complex number.

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